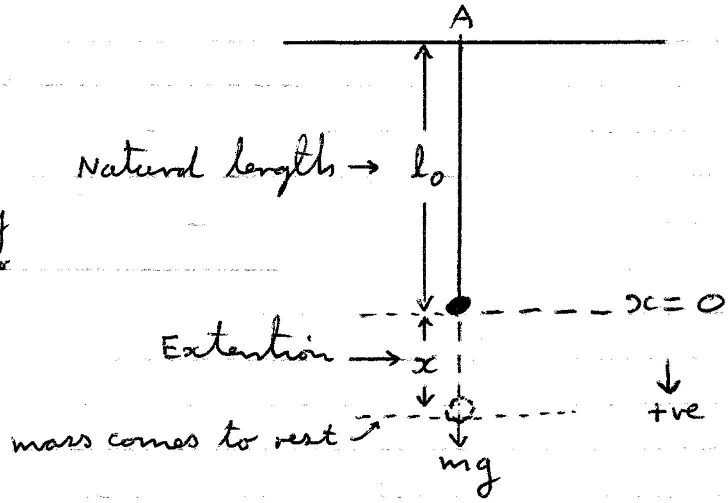
Mechanics Examples Sheet 8 – Solutions

1. K.E: kinetic energy

P.E: potential energy

E.P.E: elastic potential energy (potential energy due to elasticity of string.)



Using conservation of energy:

$$E = K.E. + P.E. + E.P.E.$$

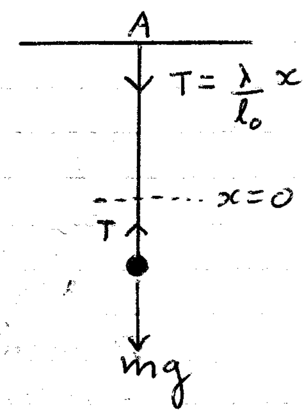
Initially : $E_i = 0 + 0 + 0$

Finally : $E_f = 0 - mgx + \frac{1}{2}\frac{\lambda}{l_0}x^2$

$$\therefore E_i = E_f \Rightarrow x\left(-mg + \frac{1}{2}\frac{\lambda}{l_0}x\right) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad \underline{x = 2mg\frac{l_0}{\lambda}}$$

Alternatively, using Newton's second law:



$$m\ddot{x} = mg - \frac{\lambda}{l_0}x$$

$$\Rightarrow m\frac{d}{dt}(\dot{x}) = mg - \frac{\lambda}{l_0}x$$

$$\Rightarrow m\dot{x}\frac{d\dot{x}}{dx} = mg - \frac{\lambda}{l_0}x$$

$$\Rightarrow \int_{\dot{x}=0}^{0} m\dot{x}\,d\dot{x} = \int_{x'=0}^{x}\left(mg - \frac{\lambda}{l_0}x'\right)dx'$$

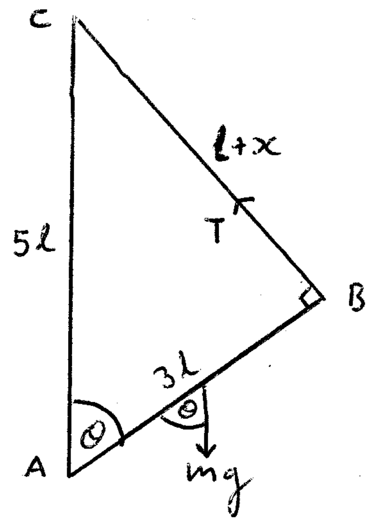
$$\Rightarrow 0 = x\left(mg - \frac{1}{2}\frac{\lambda}{l_0}x\right)$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 2mg\frac{l_0}{\lambda}$$

Both cases yield the same value for the extension. Hence, the distance below A at which the particle first comes to rest is

$$l_0 + x = \frac{l_0}{\lambda}(\lambda + 2mg)$$

2. $T = \lambda . \frac{x}{l}$



x: extension of string

From diagram:

$(l+x)^2 = (5l)^2 - (3l)^2$

$\Rightarrow x = 3l \quad \therefore \sin\theta = \frac{4}{5}$

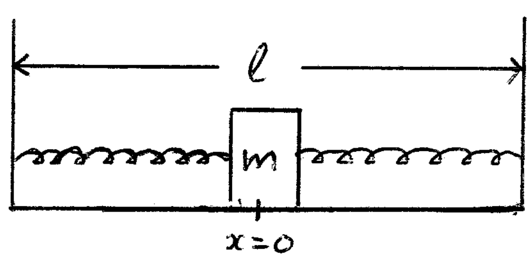
Take moments about A:

$$A) \quad mg\sin\theta\left(\frac{3l}{2}\right) = T(3l)$$

$$\Rightarrow mg\left(\frac{4}{5}\right)\left(\frac{3l}{2}\right) = \lambda\frac{x}{l}(3l) = 9l.\lambda$$

$$\Rightarrow \lambda = \frac{2}{15}mg$$

3. Initially the particle is displaced a distance x_0 from its equilibrium point at $x=0$.



We can use Hooke's law to model the behaviour of the system:

$$F(x) = -kx.$$

Combining this with N2, we get

$$m\ddot{x} = -kx$$

or

$$\ddot{x} + \omega^2 x = 0 \quad , \quad \omega = \sqrt{\frac{k}{m}}.$$

The general solution of this equation is

$$x(t) = A\cos\omega t + B\sin\omega t.$$

Also,

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t.$$

Using initial conditions

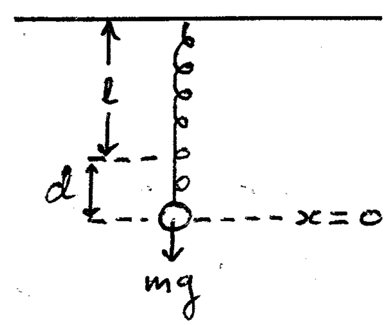
$$\dot{x}(0) = 0 \implies B = 0$$

$$x(0) = x_0 \implies A = x_0$$

yields

$$\underline{\underline{x(t) = x_0\cos\omega t}} \quad \text{with period, } \underline{\underline{\tau = \frac{2\pi}{\omega}}}$$

4. When mass m is attached and the load is in equilibrium:



$$kd = mg$$

or

$$k = \frac{mg}{d}.$$

When mass M is attached the ensuing motion is SHM and the equation of motion is

$$(m+M)\ddot{x} = -k(x-d) - (m+M)g$$

or

$$\ddot{x} + \omega^2 x = \omega^2 d - g, \qquad \omega^2 = \frac{k}{m+M}$$

or

$$\ddot{x} + \omega^2 x = -\frac{M}{m}\omega^2 d.$$

The general solution is

$$x(t) = A\cos\omega t + B\sin\omega t - \frac{M}{m}d,$$

and

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

From initial conditions: $x(0) = \dot{x}(0) = 0$, we have

$$A = (M/m)d, \quad B = 0.$$

So the position of the particle at time t is

$$x(t) = \frac{M}{m} d \left(\cos \omega t - 1 \right).$$

Thus, the amplitude is $\underline{\underline{\frac{M}{m} d}}$

the period is $\tau = \frac{2\pi}{\omega} = \underline{\underline{2\pi \sqrt{\frac{m+M}{k}}}}$

Using conservation of energy

$$E = K.E. + P.E. + E.P.E.$$

Initially : $E_i = 0 + 0 + \frac{1}{2} k d^2$

Finally : $E_f = \frac{1}{2}(m+M)\dot{x}^2 + (m+M) g x + \frac{1}{2} k (x-d)^2.$

Now, at maximum displacement $\dot{x} = 0$. Therefore,

$$E_f - E_i = 0 \Rightarrow x^2 + 2x \left[\frac{(M+m)g}{k} - d \right] = 0$$

or

$$x \left[x + 2 \frac{M}{m} d \right] = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = -2 \frac{M}{m} d.$$

Hence, the amplitude is $\underline{\underline{\frac{M}{m} d}}$